DISCUSSION OF THE CHOICE OF CONTROL POINTS IN THE METHOD OF DISCRETE VORTICES

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The method of discrete vortices is widely used for solving problems in hydrodynamics, especially in the theory of wings. The method is based on replacing a continuous vortex sheet, modeling the surface of the contact discontinuity (wing), by its discrete analog. The condition that the liquid cannot flow through the wing is satisfied at a finite number of control points. As a result the starting integral equation, corresponding to the boundary-value problem at hand, reduces to a system of linear algebraic equations for the intensity of the discrete vortices.

The carrying vortex sheet is usually uniformly divided into elements, each of which is replaced by one discrete vortex, and the control points are placed midway between the vortices. This computational scheme ensures that the solution converges for the inner part of the carrying vortex sheet [1, 2], while near its boundary an unavoidable error can arise [2, 3].

In [3, 4] it is proposed that control points be chosen taking into account the local characteristics of the carrying vortex sheet. The points at which the system of discrete vortices under study induces the same velocities as the starting vortex sheet are chosen as the control points. This choice of control points ensures that the approximate solution converges to the exact solution in the entire region, including the boundary of the wing [5]. The results in [3, 4] were obtained by dividing the vortex sheet into equal elements. In many problems, however, such a separation cannot be performed in principle. Thus in problems of stationary detached flow around a profile the velocity of shedding of free vortices from the front and back edges will in the general case be different. This results in the fact that over a chosen short time interval unequal elements of the vortex trails are shed from the edges of the profile. An analogous situation can also occur in the problems of detached flow around a wing with a finite span. For this reason it is necessary to determine the positions of the control points near the boundaries of the wing taking into account the different dimensions of the elements of the vortex sheet on the wing and in the trail.

We shall solve this problem for a thin profile in a stationary flow of an ideal incompressible fluid. In the general case vortex trails, owing to the change in the circulation of the velocity around the profile with time, will be shed from one or both edges of the profile. The problem of flow around the profile is usually formulated as a nonlinear initial- and boundary-value problem, which is solved by stepwise linearization for closely spaced times [1, 3, 6]. Let the corresponding linear boundary-value problems be solved by the method of discrete vortices with elements of the vortex sheet of the same length on the profile, while the elements of the vortex trails, forming over a time Δt, adjacent to the edges have different lengths. According to the method of discrete vortices, all elements of the vortex sheet on the profile and in the trails are occupied by discrete vortices, placed at equal distances on the profile and at the center of the elements in the trails, while the condition that the fluid not flow through the profile is satisfied at given control points. To determine the positions of the control points we shall employ, following [3, 4], the condition that in them the velocities induced by the discrete vortices and the starting vortex sheel be equal; this condition is applied for some small fixed regions on the profile and in the vicinity of the edges. Then, for the interior regions on the profile, the control points are placed midway between the vortices, and near the edges their position depends on the ratio of the lengths of the elements of the vertex sheet on the profile and in the trail.

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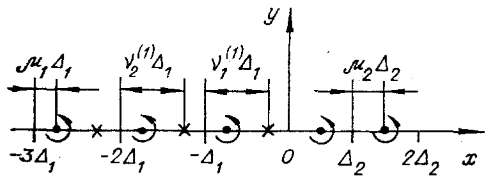

Fig. 1

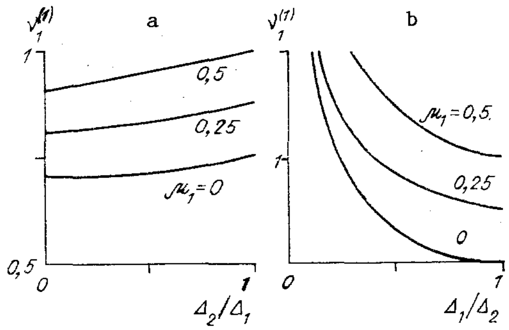

Fig. 2

We shall study the main details of the algorithm for calculating the control points in the vicinity of the back edge of the profile. Let us assume that here the vortex sheet is essentially a straight line. We introduce a Cartesian coordinate system oxy with origin in the back edge of the profile and we orient the x axis along the vortex trail. We segregate on the x axis the segment of the vortex sheet $[-\varepsilon, +\varepsilon]$ in a small neighborhood of the edge. We divide the left segment $[-\varepsilon, 0]$, corresponding to the profile, into N_1 elements of length $\Delta_1 = \varepsilon/N_1$, and the right segment $[0, +\varepsilon]$, corresponding to the trail, into N_2 elements of lengths $\Delta_2 = \varepsilon/N_2$. Let the spacing of discrete vortices on the profile be equal to $\mu_1\Delta_1$ and let the control points be placed at a distance $\nu_1^{(1)}\Delta_1$, $\nu_2^{(1)}\Delta_1$, ... from the front edge of each element. As regards the discrete vortices, modeling the vortex trail, they are located at a distance $\mu_2\Delta_2$ from the front edges of the elements (Fig. 1). Then the coordinates of the discrete vortices $x_m^{(1)}$ on the profile and $x_p^{(2)}$ in the trail are given by

$$x_m^{(1)} = \Delta_1(-m+\mu_1),\ x_p^{(2)} = \Delta_2(p-1+\mu_2),\ m=1,\ldots,N_1,$$
$$p = 1,\ldots,N_2,$$

while the coordinates of the control points are $x_{0k}^{(1)} = \Delta_1(-k+\nu_k^{(1)}),\ k=1,\ldots,N_1$.

Let us assume that in the interval $[-\varepsilon, +\varepsilon]$ the intensity of the vortex sheet $\gamma(x, t)$ is a continuous bounded function of the variable x and t, and the interval itself is so short that to a first approximation $\gamma(x, t) = \gamma(0, t)$ for all $x \in [-\varepsilon, +\varepsilon]$. Requiring that the velocities induced by the continuous vortex sheet on the segment $[-\varepsilon, +\varepsilon]$ and the system of discrete vortices be equal at the control points we arrive at a transcendental equation for determining the coefficients $\nu_k^{(1)}$:

$$\ln\frac{1+\left(k-\nu_k^{(1)}\right)/N_1}{1-\left(k-\nu_k^{(1)}\right)/N_2} + \sum_{m=1}^{N_1}\frac{1}{m-\mu_1-k+\nu_k^{(1)}} - \sum_{p=1}^{N_2}\frac{\delta}{\delta(p-1+\mu_2)+k-\nu_k^{(1)}} = 0 \tag{1}$$
$$(\delta = \Delta_2/\Delta_1,\ N_2 = N_1/\delta).$$

Equation (1) was solved in the limit $N_1 \to \infty$. The coefficients $\nu_k^{(1)}$ obtained in this manner correspond to the limiting case of unbounded growth of the elements on the profile. The position of the control points on each element of the profile thereby turns out to be independent of the number of these elements (this also happens in the usual computational scheme, when the control points are chosen strictly between the vortices). The calculation was performed for $\mu_2 = 0.5$ (the vortices in the trail are located at the center of each element) and $\mu_1 = 0$, 0.25, and 0.5 in a wide range of values of the parameter δ. The results showed that the difference in the lengths of the elements of the vortex sheet on the profile and in the trail is essentially manifested only in the change in the position of the control point, determined by the coefficient $\nu_1^{(1)}$, closest to the back edge. All other

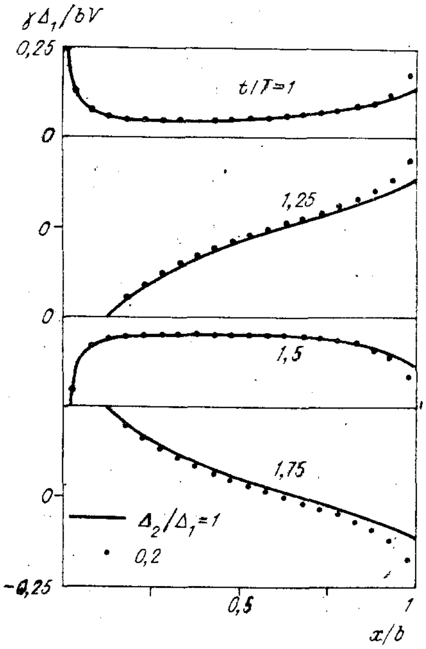

Fig. 3

control points are located practically midway between the discrete vortices for all Δ_2/Δ_1. The dependence of $\nu_1^{(1)}$ on Δ_2/Δ_1 is presented in Fig. 2 (for $0 < \Delta_2/\Delta_1 \leq 1$ (a) and for $1 \leq \Delta_2/\Delta_1 < \infty$ (b)). For sufficiently large Δ_2/Δ_1 $\nu_1^{(1)}$ becomes greater than unity, which means that the control point determined by Eq. (1) lies outside the profile. For this reason, in practical calculations, a limit must be imposed on the time step Δt, so that Δ_2/Δ_1 does not reach its critical values. For example, for $\mu_1 = 0.5$, when the vortices on the profile are chosen at the center of the elements, the condition $\nu_1^{(1)} \leq 1$ leads to the requirement $\Delta_2 \leq \Delta_1$; for $\mu_1 = 0.25$ (the vortices are located at a distance equal to one-fourth the distance from the center of the edge of each element) the condition $\Delta_2 < 2.5\ \Delta_1$, must be satisfied and for $\mu_1 = 0$ (the vortices are located at the start of the elements) $\Delta_2 < 4.3\ \Delta_1$.

As regards the calculation of the control points near the front edge in the detached flow, all results obtained remain valid with no changes, if the position of the discrete vortices and the control points on the profile (the coefficients μ_1 and $\nu_1^{(1)}$, $\nu_2^{(1)}$, ...) are measured from the back edge of the corresponding elements.

As an illustration of the results obtained we shall calculate the nonstationary flow around a plate undergoing vertical harmonic oscillations with a circular frequency ω and amplitude y_0 in the flow of an ideal incompressible fluid. For simplicity we shall confine our attention to the linear formulation of the problem, according to which the vortex trail is shed rectilinearly from the back edge and the vortices in the trail move with a constant velocity V equal to the velocity of the main (undisturbed) flow. We shall assume that the plate starts to oscillate at the time t = 0 with some fixed initial data. The algorithm for solving this problem by the method of discrete vortices is well known (see, for example, [1, 6]).

We divide the chord of the plate b into n elements of length $\Delta_1 = b/n$, each of which we replace by a discrete vortex $\Gamma_1, \ldots, \Gamma_n$, located at the center of the element ($\mu_1 = 0.5$), and we replace the vortex trail by a system of free vortices located at the center of elements of length $\Delta_2 = V\Delta t$. Setting $\Delta t = T/N$ (T is the period of oscillations of the plate ($T = 2\pi/\omega$), N is the number of computed time steps in the period T), we obtain the expression

$$\Delta_2 = \frac{n\pi}{kN}\Delta_1, \tag{2}$$

which relates the length of the elements Δ_1 and Δ_2. Here k is Strouhal's number ($k = \omega b/2V$).

We shall first perform the calculation by the standard computational scheme of the method of discrete vortices, when the control points lie strictly between the vortices. In the case

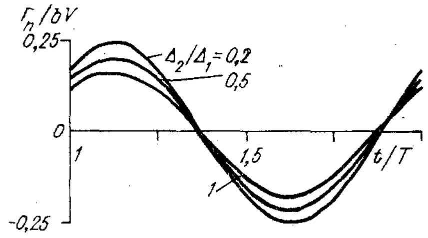

Fig. 4

at hand (μ_1 = 0.5), the control points lie on the back edges of the elements on the plate.

We note that in the algorithms for solving problems of nonstationary flow around profiles by the method of discrete vortices it is recommended that the time step Δt be chosen so that the element of the vortex trail Δ_2 formed over a time Δt be equal to the element Δ_1 on the profile [6]. This is because the discrete model of a vortex sheet permits calculating correctly the velocity at the back edge of the profile, determined for a continuous vortex sheet by a singular integral, only for $\Delta_2 = \Delta_1$. For $\Delta_2/\Delta_1 \neq 1$ there arises an error, and this error increases as the deviation of Δ_2/Δ_1 from unity increases.

We shall estimate the error in the calculation peformed by the standard scheme of the method of discrete vortices, arising with $\Delta_2 \neq \Delta_1$, for the problem of nonstationary flow around a plate. Let n = 20, k = 1, y_0 = 0.1 b, and let the number of steps N in the period T vary so that in accordance with the formula (2) the parameter δ = 0.2, 0.5, and 1.

Figure 3 shows the results of the calculation of the intensity of the vortex sheet $\gamma(x, t)$ along the chord of the oscillating plate at the times t/T = 1, 1.25, 1.5, and 1.75 for Δ_2/Δ_1 = 1 and 0.2. They show that the computational error arising owing to the differences in the lengths of the elements of the vortex sheet on the plate and in the trail is essentially manifested only in the vicinity of the back edge. The ratio Δ_2/Δ_1 affects especially strongly the accuracy of the calculation of the intensity of the vortex trail shed from the plate, which is determined by Γ_n. For this reason it is interesting to clarify the effect of Δ_2/Δ_1 on the dependence of Γ_n on the time t. The corresponding results are shown in Fig. 4. They show that the calculation by the standard scheme of the method of discrete vortices in the case $\Delta_2 \neq \Delta_1$ can lead to a large error in the calculation of the amplitude of the discrete vortex Γ_n; for Δ_2/Δ_1 = 0.2, for example, this error reaches 45%.

We shall now determine what the proposed computational scheme, in which the last control point is chosen taking into account the effect of the parameter Δ_2/Δ_1, gives, using the solution of Eq. (1). We determine the coefficient ν_n ($\nu_n = \nu_1^{(1)}$) from the formula

$$\nu_n = \begin{cases} 0.914 + 0.03\delta + 0.11\delta^2 - 0.05\delta^3, & 0 < \delta \leqslant 0.5, \\ 0.9 + 0.1\delta, & 0.5 < \delta \leqslant 1, \end{cases} \tag{3}$$

which approximates the solution of Eq. (1) for $\nu_1^{(1)}$ with μ_1 = 0.5, presented in Fig. 2. We assume that the remaining coefficients $\nu_1, \ldots, \nu_{n-1}$ are equal to unity, as in the standard scheme.

We performed the calculation for the same values of the parameters, including δ = 0.2, 0.5, and 1. The results for all $\delta < 1$ were practically identical to the data obtained for δ = 1. In other words, by changing the position of the control point at the last element of the plate according to the formula (3) the same computational accuracy as achieved with $\Delta_2 = \Delta_1$ can be preserved for $\Delta_2 \neq \Delta_1$. Thus the proposed computational scheme permits varying the time step Δt over wide limits without lowering the accuracy of the calculation in problems of nonstationary, nondetached, and detached flow around profiles.

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VORTEX-FREE PROPULSION IN AN IDEAL FLUID

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A formula for the velocity of a sphere is derived. The sphere is propelled in an ideal incompressible fluid from a state of rest by the fixed normal component of the velocity of the fluid at the permeable surface of the sphere. The fluid flow is a potential flow.

Within the framework of potential flows of an ideal incompressible fluid propulsion (self-propulsion) of bodies from a state of rest is possible owing to a periodic change in shape even though there is no propulsion force [1, 2]. V. L. Sennitskiĭ [3] and V. V. Pukhnachev [4] studied propulsion in a viscous fluid due to the fixed velocity of the fluid on the surface of the body, which was assumed to be permeable. For a sphere the optimal flows, in the sense of V. V. Pukhnachev, turned out to be potential flow. The ideal formulation of this problem is of interest. In this case the solution can be obtained simply, but the answer is nontrivial. In this connection there arises the following difficult question (which is not studied here): do close solutions exist in a fluid with low viscosity?

Let the sphere S with the radius 1 (all variables are dimensionless) be propelled from a state of rest in an ideal fluid, whose density is equal to 1, along the x-axis by the normal velocity of the fluid v_n (relative to S), which is a function of time t, given on S. Then the velocity potential of absolute motion φ satisfies the boundary condition

$$\left.\frac{\partial\varphi}{\partial n}\right|_S = v_n + \dot{x}_0\cos\theta = \sum_{k=0}^{\infty} c_k P_k(\cos\theta) + \dot{x}_0\cos\theta,$$

where $P_k(x)$ are Legendre polynomials and $P_1(x) = x$ (the flow is axisymmetric) (see Fig. 1); $\dot{x}_0$ is the velocity of the center of the sphere. We shall calculate the kinetic energy of the fluid

$$T_f = \frac{1}{2}\int\limits_{\text{outside } S} |\nabla\varphi|^2\,dx = \frac{1}{2}\sum_{k\neq 1}\frac{\alpha_k c_k}{k+1} + \frac{\alpha_1}{4}(c_1 + \dot{x}_0)^2 \quad \left(\alpha_k = \int\limits_S P_k(\cos\theta)^2\,dS,\ \alpha_1 = \frac{4\pi}{3}\right). \tag{1}$$

Let P be the total momentum inside S. We transform the equation of motion into Lagrange's form and integrate once $P + \partial T_f/\partial\dot{x}_0 = \text{const} = 0.$ Substituting here the expression (1) we obtain

$$P + (2\pi/3)(\dot{x}_0 + c_1) = 0, \tag{2}$$

whence it is obvious that the regime is optimal for $c_k = 0$ $(k \neq 1)$.

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